ECON 4130 H11

Extra exercises for no-seminar week 37

(Solutions will be put on the net at the end of the week)

Chapter 2: 54, 59, 67 (see appendix for 67c)

Chapter 3: 1, 8b

Chapter 4: 4, 6

(**Hint for 4:4.** You may need the formulas

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$
$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$)

Appendix - on using the uniform(0, 1) distribution to simulate observations from an arbitrary distribution F.

Read Proposition D and example E on page 63 in Rice. Proposition D says that, if $U \sim uniform(0,1)$, and $F(x)$ is an arbitrary *cdf*, then the rv $X = F^{-1}(U)$ has exactly this $F(x)$ as its cdf. This we can use to draw (simulate) *iid* (independent and identically distributed) observations, $x_1, x_2, \ldots, x_n$, from the *F*-distribution as follows:

- Let the computer generate a sample, $u_1, u_2, \ldots, u_n$, from the uniform(0, 1) distribution.
- Calculate $x_i = F^{-1}(u_i), \quad i = 1, 2, \ldots, n$. Then $x_1, x_2, \ldots, x_n$ is an *iid* sample drawn (simulated) from the *F*-distribution.

The proof of proposition D is just one line:

$$P(X \le x) = P(F^{-1}(U) \le x) \overset{(1)}{=} P\Big(F\big[F^{-1}(U)\big] \le F(x)\Big) \overset{(2)}{=} P\big(U \le F(x)\big) \overset{(3)}{=} F(x)$$

Explanation. Assume, for simplicity, that $F(x)$ is strictly increasing[1] (except possibly where it is exactly = 0 or = 1, i.e., so that $F^{-1}(U)$ is uniquely determined when U is different from 0 or 1. The probability that U is equal to 0 or 1 is zero, so this is no restriction.

[1] In general, $F(x)$ may be flat in certain interval. Suppose, for example, that $F(x)$ is constant $= p$ in an interval $a \le x < b$. Then $F^{-1}(p)$ is not uniquely determined (any x in $[a,b)$ could qualify). In such cases it is common practice to define $F^{-1}(p)$ conventionally as the smallest possible x satisfying $F(x) = p$. With this convention F^{-1} is always well defined, and the proof of proposition still holds.

Equality (1) follows from the equivalence: $a \le b \Leftrightarrow F(a) \le F(b)$. (If you don't see this, sketch a graph with some strictly increasing $F(x)$. Then mark *a* and *b* on the x-axis and the corresponding $F(a), F(b)$ on the y-axis.).

Equality (2) follows from noting that $F\left[F^{-1}(u)\right] = u$ for any observed value , u, of *U*. (Illustrate on your graph. Choose *u* on the y-axis.)

Equality (3) follows from the cdf of *U* being (see example A and B in Rice sec. 2.2):

$$F_U(u) = \begin{cases} 0 & \text{for } u \le 0 \\ u & \text{for } 0 < u < 1 \\ 1 & \text{for } u \ge 1 \end{cases}$$

so that $P(U \le u) = u$ for any number *u* with $0 < u < 1$.